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Algebraical confinement of coloured states

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Abstract. A mechanism of the algebraical confinement of colour based on the quaternionic structure of the space of states is investigated. A new dynamical interpretation of the colour group $SU(3)_c$ is given.

1. Introduction

The problem of quark and gluon confinement is one of the most important and mysterious questions of particle physics. Quantum chromodynamics is the candidate theory for the description of this phenomenon. The special features of the interaction maintained by the massless gluons, such as strong infrared divergencies, are believed to provide the confinement of the coloured states. However, quantum chromodynamics is not yet a complete theory and this dynamical mechanism is rather more a hypothesis than fact. It is also possible that the answer to this question is completely different and lies rather on the 'kinematical' than the dynamical level. The solution may be based on the change of the language i.e. on the interpretation of states, observables etc different from the standard one. This point of view is represented by Günaydin and Gürsey (1973a, b, 1974) Günaydin (1973, 1976), Gürsey (1974, 1976), Günaydin *et al* (1978), Ruegg (1978) and Rembieliński (1978). The scheme of Günaydin and Gürsey is based on the identification of the space of states with an octonionic Hilbert space. It was expected that the octonionic structure of the underlying Hilbert space implies the algebraical confinement of colour states. However, in this framework we have dealt with some pathologies like unobservability of two-fermion states (Kosiński and Rembieliński 1978). This follows from the over strong conditions resulting from the octonionic structure of the theory. For this reason in the papers by Rembieliński (1980a, b) a less restrictive scheme based on the quaternionic Hilbert space (QHS) was analysed. As is well known the study of quaternion quantum mechanics was undertaken by a number of authors (see, for example, Emch 1963a, b, Finkelstein *et al* 1962, 1963a, b, Jauch 1968a, b). However, the physical content of the quaternionic theories depends strongly on the appropriate association of the pure states with rays and definition of the tensor product of the QHS.

The purpose of this paper is to give a systematical analysis of the quaternionic theory proposed in the papers by Rembieliński (1980a, b). We start with a brief survey of the QHS formalism in § 2. In § 3 we investigate the question of association of the pure states with rays (projectors) in QHS. This problem follows from the possibility of coexistence of three different geometrical structures in QHS: real $\mathbb{R}$, complex $\mathbb{C}$ and quaternionic $\mathbb{Q}$ and consequently with three possible projective geometries. The requirement of the uniqueness for the construction method of the multiparticle states leads to the preference of complex and real geometries. The real case seems to be rather unattractive because of well known reasons (Stuckelberg 1960, Stuckelberg and Guenin 1961 a, b, Jauch 1968a, b; for the experimental test see Peres 1979, Rembieliński 1979). Therefore the pure states are identified with the complex rays in QHS. The tensor product definition of QHS (Rembieliński 1980a, b) is investigated in detail (see §§ 3, 4). It is remarkable that the form of the tensor product of QHS is in general, dependent on the structure of the symmetry group of the theory. The class of the groups and representations which are admissible with respect to the quaternionic structure (Rembieliński 1980a, b) is extended to the nonlinear realisation case. Section 5 is devoted to the discussion of physical consequences of QHS formalism like splitting of the space of states on the observable and unobservable sectors, the superselection rule caused by this fact, a new dynamical interpretation of the colour group and its consequences.

2. Some mathematical notions

In this section a brief survey of the QHS formalism is given.

2.1. *The quaternion field*

As is well known, the Frobenius theorem states that the only finite dimensional division algebras over the field of real numbers are: real numbers themselves, complex numbers and quaternions. Moreover, the real and complex numbers form the subfields of the quaternion field. Thus the quaternions occupy an exceptional position in mathematics.

A real quaternion $a = e_0 a_0 + \boldsymbol{e a} \equiv e_\mu a_\mu$ (for the convention see Rembieliński (1980a, b)) can be also represented in the symplectic form $a = e_0 A_0 + e_1 A_1 \equiv e_\alpha A_\alpha$ where $A_0 = e_0 a_0 + e_3 a_3$ and $A_1 = e_0 a_1 - e_3 a_2$. Note that the A_α belong to the subfield $\mathbb{C}(e_0, e_3)$ of $\mathbb{Q}$ isomorphic to the field of complex numbers ($e_0 \sim 1, e_3 \sim \sqrt{-1}$). All other choices of $\mathbb{C} \subset \mathbb{Q}$ are equivalent to $\mathbb{C}(e_0, e_3)$ because SO(3), the automorphism group of $\mathbb{Q}$, acts transitively on the unit sphere. In the following we use the notions of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{Q}$ conjugation of $\mathbb{Q}$:

$C_{\mathbb{R}} \equiv$ identity

$C_{\mathbb{C}}$, complex conjugation (automorphism) $e_0^{C_{\mathbb{C}}} \equiv e_0^* = e_0$, $e_1^* = e_1$, $e_2^* = -e_2$, $e_3^* = -e_3$

$C_{\mathbb{D}}$, automorphism defined by $e_0^{C_{\mathbb{D}}} \equiv \tilde{e}_0 = e_0$, $\tilde{e}_1 = -e_1$, $\tilde{e}_2 = -e_2$, $\tilde{e}_3 = e_3$

$C_{\mathbb{Q}}$, quaternionic conjugation (anti-automorphism) $e_0^{C_{\mathbb{Q}}} \equiv \bar{e}_0 = e_0$, $\bar{e}_k = -e_k$, $k = 1, 2, 3$.

By definition $\bar{C}_{\mathbb{R}} = C_{\mathbb{Q}}$, $\bar{C}_{\mathbb{C}} = C_{\mathbb{D}}$, $\bar{C}_{\mathbb{Q}} = C_{\mathbb{R}}$.

2.2. *Quaternionic Hilbert space*

As was mentioned in the introduction, the quaternionic Hilbert space has been used in a number of papers. In this article we formulate the QHS axioms in a fashion which is equivalent but independent of the choice of specific geometry (real, complex or quaternionic). A quaternionic Hilbert space $\mathscr{H}_{\mathbb{Q}}$ is a linear vector space over the field of quaternions $\mathbb{Q}$. In this space we introduce the $\mathbb{A}$ valued ($\mathbb{A}=\mathbb{R},\mathbb{C}$ or $\mathbb{Q}$) scalar product $(f, g)_{\mathbb{A}}$ defined for all f, g in $\mathscr{H}_{\mathbb{Q}}$ by the axioms

$$(f, g+h)_{\mathbb{A}}=(f, g)_{\mathbb{A}}+(f, h)_{\mathbb{A}}$$

$$(f, g)_{\mathbb{A}}^{C_{\mathbb{A}}}=(g, f)_{\mathbb{A}}$$

$$(f, fa)_{\mathbb{A}}=\tfrac{1}{2}(a+a^{\bar{C}_{\mathbb{A}}})|f|_{\mathbb{A}}^{2} \qquad \text{where } |f|_{\mathbb{A}}^{2}=(f, f)_{\mathbb{A}}\geqslant 0$$

and $|f|_{\mathbb{A}}=0$ is equivalent to $f=0$

$$(f, g\alpha)_{\mathbb{A}}=(f, g)_{\mathbb{A}}\alpha \text{ for } \alpha\in\mathbb{A}.$$

In addition we postulate the completeness of $\mathscr{H}_{\mathbb{Q}}$ i.e. any Cauchy sequence $\{h_n\}$, $h_n\in\mathscr{H}_{\mathbb{Q}}$, defines a unique limit $h\in\mathscr{H}_{\mathbb{Q}}$ such that $\lim_{n\to\infty}|h_n-h|_{\mathbb{A}}=0$. The above axioms are consistent with the properties of the $\mathbb{A}$-scalar product in the quaternion algebra (see the appendix). It is important that a fixed $\mathbb{A}$-geometry induces all others via the formula

$$(f, g)_{\mathbb{A}}=\tfrac{1}{2}\sum_{\mu=0}^{3}\{\tfrac{1}{2}(e_\mu+e_\mu^{C_{\mathbb{B}}})(f\tfrac{1}{2}(e_\mu+e_\mu^{C_{\mathbb{B}}}), g)_{\mathbb{B}}+[\tfrac{1}{2}(e_\mu+e_\mu^{C_{\mathbb{B}}})(f\tfrac{1}{2}(e_\mu+e_\mu^{C_{\mathbb{B}}}), g)_{\mathbb{B}}]^{\bar{C}_{\mathbb{A}}}\} \qquad (1)$$

where $\mathbb{A}, \mathbb{B}=\mathbb{R}, \mathbb{C}$ or $\mathbb{Q}$. Using the above relation it is immediately obvious that the weak topologies induced by $\mathbb{A}$-scalar products are equivalent. Similarly all strong topologies are equivalent too. In particular $|f|_{\mathbb{R}}=|f|_{\mathbb{C}}=|f|_{\mathbb{Q}}\equiv|f|$ for every $f\in\mathscr{H}_{\mathbb{Q}}$.

2.3. *Linear manifolds and linear operators*

A subset $M_{\mathbb{A}}$ of the quaternionic Hilbert space $\mathscr{H}_{\mathbb{Q}}$ is called an $\mathbb{A}$-linear manifold if $f, g\in M_{\mathbb{A}}$ implies $f\alpha+g\beta\in M_{\mathbb{A}}$ where α and β belong to the field $\mathbb{A}$. We remark that the notion of the closed $\mathbb{A}$-linear manifold and the subspace of $\mathscr{H}_{\mathbb{Q}}$ are equivalent only for $\mathbb{Q}$-linear manifolds. The $\mathbb{A}$-linear operator L is defined as an $\mathbb{A}$-linear mapping of the manifold $M_{\mathbb{A}}\subset\mathscr{H}_{\mathbb{Q}}$ (domain) into $\mathscr{H}_{\mathbb{Q}}$. The $\mathbb{A}$-adjoint operator $L^{\dagger}$ is defined by the standard formula $(Lf, g)_{\mathbb{A}}=(f, L^{\dagger}g)_{\mathbb{A}}$. As usual we can define $\mathbb{A}$-Hermitian (eventually $\mathbb{A}$-self-adjoint) and $\mathbb{A}$-unitary operators by the rules

$$(Hf, g)_{\mathbb{A}}=(f, Hg)_{\mathbb{A}} \qquad (Uf, g)_{\mathbb{A}}=(f, U^{-1}g)_{\mathbb{A}}$$

and appropriate choice of the domain. The notions of the norm, continuity, boundedness etc of operators can be introduced in a standard way. In particular the projectors on the closed $\mathbb{A}$-linear manifolds are bounded $\mathbb{A}$-self-adjoint. The $\mathbb{A}$-antilinear operator K is defined by

$$K(f\alpha+g\beta)=(Kf)\alpha^{C_{\mathbb{A}}}+(Kg)\beta^{C_{\mathbb{A}}}$$

where $\alpha, \beta\in\mathbb{A}$. Following Jauch (1968a) we define an $\mathbb{A}$-semi-linear operator T by the formula

$$T(f\alpha+g\beta)=(Tf)\alpha^{s}+(Tg)\beta^{s}$$

where $\alpha, \beta\in\mathbb{A}$ and s is an automorphism of $\mathbb{A}$.

Finally, we note that $\mathbb{Q}$ linearity (Hermiticity, unitarity) implies $\mathbb{C}$ and $\mathbb{R}$ linearity, $\mathbb{C}$ linearity implies $\mathbb{R}$ linearity. However the converse statement is not true.

2.4. *Complex decomposition of* QHS

Analogous to the complex (symplectic) decomposition of quaternions from $\mathbb{Q}$ there exists the corresponding decomposition of vectors in $\mathscr{H}_{\mathbb{Q}}$. In order to express this in a formula let us implement the conjugations $C_{\mathbb{C}} \equiv C$ and $C_{\mathbb{D}} \equiv D$ by the appropriate involutions in $\mathscr{H}_{\mathbb{Q}}$ as follows

$$\begin{aligned} &C_{\mathbb{A}}(f+g) = C_{\mathbb{A}}f + C_{\mathbb{A}}g \qquad C_{\mathbb{A}}(fq) = (C_{\mathbb{A}}f)q^{C_{\mathbb{A}}} \\ &C_{\mathbb{A}}^2 f = f \qquad (C_{\mathbb{A}}f, C_{\mathbb{A}}g)_{\mathbb{R}} = (f, g)_{\mathbb{R}}. \end{aligned} \tag{2}$$

Here $\mathbb{A} = \mathbb{C}$ or $\mathbb{D}$. From equations (1)–(2) we see that C is $\mathbb{C}$ anti-unitary, D is $\mathbb{C}$ unitary and $\mathbb{C}$ Hermitian. Both C and D are $\mathbb{R}$ unitary and $\mathbb{R}$ Hermitian. We shall denote them by $Cf \equiv f^*$, $Df \equiv \tilde{f}$ respectively. Now, the $\mathbb{C}$ projectors $\Pi_{\pm} = \frac{1}{2}(I \pm D)$ split $\mathscr{H}_{\mathbb{Q}}$ into two $\mathbb{C}$-linear, mutually $\mathbb{C}$-orthogonal manifolds $M_{\pm}$. Thus $\mathscr{H}_{\mathbb{Q}} = M_+ + M_-$ and appropriately every vector $f \in \mathscr{H}_{\mathbb{Q}}$ has the decomposition $f = f_+ + f_-$. Defining $f_+ = f_0 e_0$ and $f_- = f_1^* e_1$ we have

$$f = f_0 e_0 + f_1^* e_1 \tag{3a}$$

or in a realisation of QHS as a space of $\mathbb{Q}$-valued functions

$$f = e_0 f_0 + e_1 f_1 \equiv e_\alpha f_\alpha . \tag{3b}$$

Note that f_α, $\alpha = 0, 1$, satisfy the 'pure complexity' condition $\tilde{f}_\alpha = f_\alpha$ and therefore in a concrete realisation they are $\mathbb{C}$ valued. For this reason the decomposition (3) is called the symplectic representation of vectors from the QHS. Using the formula $(u, he_1)_{\mathbb{C}} = -(h, ue_1)_{\mathbb{C}}$ obtained from the geometrical axioms of the QHS, we can rewrite the $\mathbb{C}$-scalar product in the form

$$(f, g)_{\mathbb{C}} = (f_0, g_0)_{\mathbb{C}} + (f_1, g_1)_{\mathbb{C}}. \tag{4}$$

Note that this form is invariant under $U(2)_c$ unitary group transformations of the components f_α and g_α.

2.5. *The quaternionic group*

It is easy to see that eight operators $\pm E_\mu$ defined by

$$\pm E_\mu f = f(\pm e_\mu) \tag{5}$$

form a discrete group—the so-called quaternionic group (Hamermesh 1962). As is well known this group possesses one faithful (two-dimensional) and three homomorphic (one-dimensional, abelian) representations (except for a trivial one). Furthermore, the direct product of the odd number of the faithful representations contains only faithful ones in its decomposition while the even product decomposes on the abelian homomorphic representations.

The action (5) of the operators E_μ can be realised in the symplectic representation (3) as the (left) action on the components $f_\alpha(e_0, e_3)$. The explicit form of $(E_\mu)_{\alpha\beta}$ is as follows:

$$E_0 = I \qquad E_1 = -\varepsilon C \qquad E_2 = e_3 \varepsilon C \qquad E_3 = e_3 I. \tag{6}$$

Here $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and C denotes the above introduced operator of complex conjugation.

2.6. *Connection with the complex Hilbert space*

If we restrict ourselves to the complex geometry (complex scalar product) and to the multiplication of vectors by complex numbers from $\mathbb{C}$ then the geometrical and topological structure of $\mathscr{H}_{\mathbb{Q}}$ is isomorphic to the corresponding one of the complex Hilbert space (see, for example, Rembieliński 1980a, b). In fact, if the field of scalars is restricted to $\mathbb{C}$ then the postulates of the QHS with complex scalar product have the standard complex Hilbert space form. The definitions of the $\mathbb{C}$-linear manifold, $\mathbb{C}$-linear, Hermitian and unitary operators also coincide. Taking into account the symplectic decomposition of $\mathscr{H}_{\mathbb{Q}}$ we see that with every vector $f = e_\alpha f_\alpha (e_0, e_3) \in \mathscr{H}_{\mathbb{Q}}$ can be associated a complex vector $f = \binom{f_0(1,\mathrm{i})}{f_1(1,\mathrm{i})}$ belonging to the complex Hilbert space $\mathscr{H}_{\mathbb{C}}$. Thus $\mathscr{H}_{\mathbb{C}} = \mathscr{H}_+ + \mathscr{H}_-$ where the complex subspaces $\mathscr{H}_\pm$ are associated with the $\mathbb{C}$-manifolds $M_\pm$ of $\mathscr{H}_{\mathbb{Q}}$. The quaternionic units e_0 and e_3 are represented in $\mathscr{H}_{\mathbb{C}}$ by 1 and $\mathrm{i} = \sqrt{-1}$. The scalar product in $\mathscr{H}_{\mathbb{C}}$ has the form $(f, g) = (f_0, g_0) + (f_1, g_1)$. Completeness of $\mathscr{H}_{\mathbb{Q}}$ implies completeness of $\mathscr{H}_{\mathbb{C}}$. Furthermore, in $\mathscr{H}_{\mathbb{C}}$ the multiplication by quaternions can be implemented by E_μ via the formula

$$fq = \sum_{\mu=0}^{3} E_\mu f q_\mu \tag{7}$$

where $q = e_\mu q_\mu$ and as follows from equation (6) E_μ are represented by

$$E_0 = I \qquad E_1 = -\mathrm{i}\sigma_2 C \qquad E_2 = -\sigma_2 C \qquad E_3 = \mathrm{i}I. \tag{8}$$

Here C denotes the complex conjugation in $\mathscr{H}_{\mathbb{C}}$. The involution D is represented by σ_3 (σ_k are the Pauli matrices). The vectors $f = \binom{f_0}{f_1}$ form the two-dimensional representation of the above mentioned $\mathrm{U}(2)_{\mathrm{c}}$ group.

The above results can be derived with more mathematical rigour by considering the space of the $\mathbb{C}$-linear, continuous functionals over QHS. Because the QHS postulates reduce to the complex Hilbert space ones if we restrict ourselves to the multiplication by complex scalars and to the complex scalar product, the Hahn–Banach and Riesz theorems (see, for example, Yosida 1974) hold. Therefore with every vector $f \in \mathscr{H}_{\mathbb{Q}}$ there is associated a $\mathbb{C}$-linear continuous functional $\mathscr{F}$ by the formula $\mathscr{F}(g) = (f, g)_{\mathbb{C}}$. According to the standard procedure, in the pre-Hilbert space of the $\mathbb{C}$-linear functionals we define the scalar product of two functionals $\mathscr{F}$ and $\mathscr{G}$ as $(\mathscr{F}, \mathscr{G}) = (g, f)_{\mathbb{C}}$. The multiplication by quaternions we introduce by the formula

$$(\mathscr{F}q)(g) = (fq^*, g)_{\mathbb{C}}$$

for every $g \in \mathscr{H}_{\mathbb{Q}}$. It is easy to see that this space is isomorphic to the $\mathscr{H}_{\mathbb{C}}$ defined above.

2.7. *The role of the $U(2)_c$ group*

At the end of § 2.4 it was mentioned that the $\mathbb{C}$-scalar product in QHS is invariant with respect to transformations of the $\mathrm{U}(2)_{\mathrm{c}}$ group. Under action of this group the components f_α of f behave as a $\mathrm{U}(2)_{\mathrm{c}}$ doublet. From equation (1) it follows that this group also leaves the $\mathbb{R}$-scalar product invariant while for $\mathrm{SU}(2)_{\mathrm{c}} \subset \mathrm{U}(2)_{\mathrm{c}}$ the $\mathbb{Q}$-scalar product remains unaffected. Moreover $\mathrm{SU}(2)_{\mathrm{c}}$ define the so-called collinear quaternionic multiplication (Finkelstein *et al* 1963a, b) by the identification σ_0 and $-\mathrm{i}\boldsymbol{\sigma}$ (in $\mathscr{H}_{\mathbb{C}}$ notation) with the quaternionic units e_0 and $\boldsymbol{e}$ acting from the left. Furthermore, $\mathrm{SU}(2)_{\mathrm{c}}$ is connected with the multiplicative group $\mathrm{SU}(2)_{\mathbb{Q}}$ of unit quaternions, acting from the right, by the equivalence transformation $T = (1/\sqrt{2})(\sigma_0 + \mathrm{i}\sigma_3)\begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$. Note that the quaternionic group (6) belongs to $\mathrm{SU}(2)_{\mathbb{Q}}$.

Summarising this section it is sensible to stress that in view of equation (1) and the equivalence of all weak (strong) QHS topologies a concrete choice of $\mathbb{A}$-scalar product

($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$) in the axioms of the QHS is immaterial from the formal point of view. Nevertheless, as we see below, this question is important for the physical applications of the QHS formalism. Furthermore, we take notice that the language of the complex Hilbert space $\mathscr{H}_{\mathbb{C}}$ defined in § 2.6 is more useful for formulating the quantum mechanical notions and, as we see, for the explicit definition of the tensor product of QHS. However, the more genuinely quaternionic notions, like those connected with $\mathbb{Q}$-geometrical structure of QHS and preservation of the algebraical one in the tensor product, are less natural on this ground†. For this reason we shall use interchangeably both ($\mathscr{H}_{\mathbb{Q}}$ and $\mathscr{H}_{\mathbb{C}}$) descriptions.

3. Tensor product of QHS and physical states

The problem of the tensor product of QHS was analysed by Finkelstein *et al* (1962) under the assumption that the pure states are identified with quaternionic rays‡. It was claimed that there does not exist a satisfactory definition of the product states—a unique tensor product of QHS cannot be defined. In the present paper we investigate this question from a different point of view (see also Rembieliński 1980a, b). Let us denote a product vector by $f^1 \times f^2 \times \ldots \times f^N$ where $f^k \in \mathscr{H}^k_{\mathbb{Q}}$. As usual, the distributivity condition is assumed

$$f^1 \times \ldots \times (h^k + g^k) \times \ldots \times f^N = f^1 \times \ldots \times h^k \times \ldots \times f^N + f^1 \times \ldots \times g^k \times \ldots \times f^N \tag{9}$$

for every $k = 1, 2, \ldots, N$. It is obvious that the physical content of the theory is determined by the identification of (pure) states with rays in the QHS. However, three possibilities exist because we can associate states with quaternionic, complex or real rays. Let us assume that states are identified with $\mathbb{A}$-rays ($\mathbb{A} = \mathbb{R}, \mathbb{C}$ or $\mathbb{Q}$). Then the vectors $f^1\alpha^1, f^2\alpha^2 \ldots$ and $f^N\alpha^N$, where $\alpha^k \in \mathbb{A}$, determine the same states as f^1, f^2, $\ldots, f^N$, respectively. Therefore, the product vectors $f^1\alpha^1 \times f^2\alpha^2 \times \ldots \times f^N\alpha^N$ and $f^1 \times f^2 \times \ldots \times f^N$ must determine the same state in $\mathscr{H}^1_{\mathbb{Q}} \times \mathscr{H}^2_{\mathbb{Q}} \times \ldots \times \mathscr{H}^N_{\mathbb{Q}}$. Thus under the condition that the product space is a linear vector space and states, as formerly, are identified with rays, we must have

$$f^1 \times f^2 \times \ldots \times f^k\alpha \times \ldots \times f^N = (f^1 \times f^2 \times \ldots \times f^k \times \ldots \times f^N)\alpha^{\tau_k} \tag{10}$$

for every $k = 1, 2, \ldots, N$. Here $\alpha \in \mathbb{A}$ while α^{τ_k} belongs to the field of scalars of the product space. From equations (9)–(10) we obtain

$$(\alpha\beta)^{\tau_k} = \alpha^{\tau_k}\beta^{\tau_k} \qquad (\alpha + \beta)^{\tau_k} = \alpha^{\tau_k} + \beta^{\tau_k} \tag{11}$$

i.e. the subfield $\mathbb{A}^{\tau_k}$ of the field of scalars of the product space is homomorphic to $\mathbb{A}$. However, because of the Frobenius theorem (see § 2.1) this homomorphism must be an isomorphism. Furthermore, from equation (10) we have

$$f^1 \times \ldots \times f^k(\alpha\beta) \times \ldots \times f^N = (f^1 \times \ldots \times f^k \times \ldots \times f^N)(\alpha\beta)^{\tau_k}$$

† An analogous situation arises if the structure and the tensor product of the complex Hilbert spaces are described with the help of the real Hilbert space language.

‡ A real, complex or quaternionic ray is defined as the equivalence class of vectors of the form $f\alpha$, where $f \in \mathscr{H}_{\mathbb{Q}}$ is fixed while $\alpha \in \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ respectively.

and simultaneously

$$f^1\times\ldots\times f^k(\alpha\beta)\times\ldots\times f^i\times\ldots\times f^N=f^1\times\ldots\times f^k\alpha\times\ldots\times f^i\beta^{\tau_k\tau_i^{-1}}\times\ldots\times f^N$$
$$=(f^1\times\ldots\times f^k\times\ldots\times f^i\times\ldots\times f^N)\beta^{\tau_k}\alpha^{\tau_k}.$$

Therefore

$$(\alpha\beta)^{\tau_k}=(\beta\alpha)^{\tau_k}. \tag{12}$$

Because τ_k is an isomorphism then from equation (12) it follows that the field $\mathbb{A}$ must be abelian i.e. $\mathbb{A}=\mathbb{C}$ or $\mathbb{R}$. Consequently the pure states can be associated with complex or real rays only. In the following we restrict ourselves to the complex case. The preference for the complex variant of quantum theory follows from the well known fact that for the real case the complex structure must also be introduced to guarantee the correct implementation of the classical Poisson bracketts (Jauch 1968a, b, Mackey 1963, 1978).

Now, with the help of the relation (10) the multiplication by complex numbers is defined in the product space. In order to determine the multiplication by quaternions let us consider the representation of the quaternionic group in $\mathscr{H}^1_{\mathbb{Q}}\times\ldots\times\mathscr{H}^N_{\mathbb{Q}}$

$$D(\pm E_\mu)(f^1\times\ldots\times f^N)\equiv\pm E_\mu(f^1\times\ldots\times f^N)=(\pm E_\mu f^1)\times\ldots\times(\pm E_\mu f^N). \tag{13}$$

Note that the representation of $SU(2)_{\mathbb{Q}}$ is defined as

$$D(q)(f^1\times\ldots\times f^N)=f^1q\times\ldots\times f^Nq \qquad |q|=1.$$

As was mentioned in § 2.5, for N odd we obtain a faithful representation of $\{\pm E_\mu\}$ and consequently it is possible to define the multiplication by quaternions as follows

$$(f^1\times\ldots\times f^N)q=\sum_{\mu=0}^{3}E_\mu(f^1\times\ldots\times f^N)q_\mu. \tag{14}$$

Here $q=e_\mu q_\mu$, $q_\mu\in\mathbb{R}$ and N is odd. Note that because of equation (12) it is impossible to define multiplication by quaternions as

$$f^1\times\ldots\times f^kq\times\ldots\times f^N\equiv(f^1\times\ldots\times f^N)q^{\tau_k}$$

for every $k=1,2,\ldots,N$. It is also important that $(f^1\times\ldots\times f^N)q$ (see equation (14)) is not equivalent to $f^1q\times\ldots\times f^Nq$ ($D(q)$ define a representation of $\mathbb{Q}$ as the multiplicative group rather than the field one).

On the other hand, for N even the representation of $\{\pm E_\mu\}$ is abelian (homomorphic) and thus it is impossible to introduce the multiplication of vectors by quaternions. Therefore in this case the product space is a complex linear vector space†.

In § 2.7 the role of the group $SU(2)_c$ in the structure of QHS was stressed. In particular, the group $SU(2)_{\mathbb{Q}}$ of unit quaternions connected with $SU(2)_c$ by the similarity

† Following Bourbaki (1974) the tensor product of modules defined over a non-commutative ring A is introduced as the product between a right and left A module. As a result we obtain a Z module (Z is the set of integers). This definition can be generalised to the case of a bi-module: If $M_{(AB)}$ is an (A, B) bi-module i.e. it is a left A module and right B module, then

$$M_{(AB)}\otimes_B M_{(BC)}\equiv M_{(AC)}$$

is an (A, C) bi-module. Because the QHS can be equipped with the $(\mathbb{R}, \mathbb{Q})$ bi-module structure, the resulting space is real (even N) or quaternionic (odd N). However, we must modify this definition because in our formalism the pure states are the complex rays i.e. the even product of the QHS should be a complex space.

transformation T (§ 2.7) remains unaffected the quaternionic rays in the QHS or equivalently every $\mathbb{Q}$ manifold is invariant under the action of this group. In order to preserve the quaternionic structure as much as possible we demand that in the product space the role of $SU(2)_\mathbb{Q}$ is the same as in the QHS. Precisely, this denotes that the representation of $SU(2)_\mathbb{Q}$ in $\mathscr{H}^1_\mathbb{Q} \times \ldots \times \mathscr{H}^N_\mathbb{Q}$ should commute with every $\mathbb{C}$-projector† Π which commutes with the operations E_μ defined in equation (13). The inverse statement is also true because E_μ belong to $SU(2)_\mathbb{Q}$. In other words, the quaternionic groups $\{\pm E_\mu\}$ and $SU(2)_\mathbb{Q}$ should have the common commutant in the product space, namely a common set of operators which is spanned by the intertwining operators of the (reducible) representation of $\{\pm E_\mu\}$ or $SU(2)_\mathbb{Q}$.

Now, for clarity, we reformulate the above conditions in the complex Hilbert space language (§ 2.6). Firstly, because of the equivalence of the $SU(2)_\mathbb{Q}$ and $SU(2)_c$ descriptions we can formulate the last condition as follows. The representations of $SU(2)_c$ ($\exp i\boldsymbol{\varphi J}$) and the collinear quaternionic group ($\pm\varepsilon_k = \exp(\pm i\pi J_k)$, $\pm\varepsilon_0 = \pm I$) should have a common commutant in the product space. Thus for even products of QHS the only admissible representations of $SU(2)_c$ form a direct sum of scalars (**1**) because in this case the representation of the (collinear) quaternionic group is abelian (§ 2.5). For odd products of QHS the representation of the quaternionic group is a direct sum of two-dimensional faithful irreducible representations and consequently the admissible representations of $SU(2)_c$ contain the SU(2) doublets only. Secondly, the most general product of $\mathscr{H}_\mathbb{Q}$ satisfying the other conditions discussed above, like equations (9)–(10) for $\mathbb{A} = \mathbb{C}$, coincides with the tensor product of the complex Hilbert spaces. Therefore the most general product of QHS satisfying all the requirements discussed should be defined as follows (Remblieliński 1980a, b)

$$\mathscr{H}^1_\mathbb{Q} \times \ldots \times \mathscr{H}^N_\mathbb{Q} = \Pi(\mathscr{H}^1_\mathbb{Q} \underset{c}{\otimes} \ldots \underset{c}{\otimes} \mathscr{H}^N_\mathbb{Q}). \tag{15}$$

Here $\underset{c}{\otimes}$ denotes the standard tensor product while Π projects on the space of the admissible representations of $SU(2)_c$. The complex scalar product in $\mathscr{H}^1_\mathbb{Q} \times \ldots \times \mathscr{H}^N_\mathbb{Q}$ is induced by the scalar product in $\mathscr{H}^1_\mathbb{Q} \underset{c}{\otimes} \ldots \underset{c}{\otimes} \mathscr{H}^N_\mathbb{Q}$. It satisfies the desired inequality $|f^1 \times \ldots \times f^N| \leqslant |f^1| \times |f^2| \times \ldots \times |f^N|$ (Jauch 1968a, b). With respect to this scalar product the obtained pre-Hilbert space can be completed. For odd N the other scalar products can be derived from formula (1).

It is remarkable that our definition (15) satisfies the naturality condition. In a concrete realisation of QHS, as a space of $\mathbb{Q}$-valued functions, the product vectors are linear combinations of the quaternionic products of vectors belonging to the QHS. For the product $\mathscr{H}^1_\mathbb{Q} \times \mathscr{H}^2_\mathbb{Q} = \Pi(\mathscr{H}^1_\mathbb{Q} \underset{c}{\otimes} \mathscr{H}^2_\mathbb{Q})$, for example, the operator Π projects on the singlet of $SU(2)_c$: $\mathbf{2} \times \mathbf{2} = \Pi(\mathbf{2} \otimes \mathbf{2}) = \Pi(\mathbf{3} \oplus \mathbf{1}) = \mathbf{1}$. Thus the product vectors have the form

$$\chi = \frac{1}{\sqrt{2}} \varepsilon_{\alpha\beta} f^1_\alpha f^2_\beta = -\frac{1}{2\sqrt{2}} e_1(\bar{f}^1 f^2 - \widetilde{\bar{f}^1 f^2}) \tag{16a}$$

where $f^k \in \mathscr{H}^k_\mathbb{Q}$ and $\varepsilon_{\alpha\beta}$ is given in equation (6). The scalar product of two vectors of the form (16a) becomes

$$(\chi, \psi)_\mathbb{C} = \tfrac{1}{2}\varepsilon_{\alpha\alpha'}\varepsilon_{\beta\beta'}(f^1_\alpha, g^1_{\alpha'})_\mathbb{C}(f^2_\beta, g^2_{\beta'})_\mathbb{C}. \tag{16b}$$

† For the odd product of the QHS this condition reduces to the obvious requirement of commutativity between $SU(2)_\mathbb{Q}$ and every $\mathbb{Q}$ projector.

For the product $\mathscr{H}_{\mathbb{Q}}^1 \times \mathscr{H}_{\mathbb{Q}}^2 \times \mathscr{H}_{\mathbb{Q}}^3$ of three copies of the QHS, the resulting space, is essentially quaternionic because $\mathbf{2}\times\mathbf{2}\times\mathbf{2} = \Pi(\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}) = \Pi(\mathbf{4}\oplus\mathbf{2}\oplus\mathbf{2}) = \mathbf{2}\oplus\mathbf{2}$. The product vectors can be represented as follows

$$\varphi_{\alpha\beta\gamma} = \Pi_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'} f_{\alpha'}^1 f_{\beta'}^2 f_{\gamma'}^3$$

where

$$\Pi_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'} = \delta_\alpha^{\alpha'}\delta_\beta^{\beta'}\delta_\gamma^{\gamma'} - \tfrac{1}{6}(\delta_\alpha^{\alpha'}\delta_\beta^{\beta'}\delta_\gamma^{\gamma'} + \delta_\alpha^{\beta'}\delta_\beta^{\gamma'}\delta_\gamma^{\alpha'} + \delta_\alpha^{\gamma'}\delta_\beta^{\alpha'}\delta_\gamma^{\beta'} + \delta_\alpha^{\beta'}\delta_\beta^{\alpha'}\delta_\gamma^{\gamma'} + \delta_\alpha^{\alpha'}\delta_\beta^{\gamma'}\delta_\gamma^{\beta'} + \delta_\alpha^{\gamma'}\delta_\beta^{\beta'}\delta_\gamma^{\alpha'}).$$

The scalar product becomes

$$(\varphi,\chi)_{\mathbb{C}} = \sum_{\substack{\alpha\beta\gamma\\ \alpha'\beta'\gamma'}} \Pi_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'} (f_\alpha^1, g_{\alpha'}^1)_{\mathbb{C}} (f_\beta^2, g_{\beta'}^2)_{\mathbb{C}} (f_\gamma^3, g_{\gamma'}^3)_{\mathbb{C}}.$$

In this case (in general for odd N) the product vectors and its scalar product can be rewritten in the symplectic form (3)–(4) with the help of the generator J_3 of $\mathrm{SU}(2)_c$. In the general case, if the product vectors are of the form

$$\varphi_{\alpha_1\ldots\alpha_N} = \Pi_{\alpha_1\ldots\alpha_N}^{\beta_1\ldots\beta_N} f_{\beta_1}^1 \ldots f_{\beta_N}^N \tag{17a}$$

then the scalar product reads

$$(\varphi,\chi)_{\mathbb{C}} = \sum_{\substack{\alpha_1\ldots\alpha_N\\ \beta_1\ldots\beta_N}} \Pi_{\alpha_1\ldots\alpha_N}^{\beta_1\ldots\beta_N} (f_{\alpha_1}^1, g_{\beta_1}^1)_{\mathbb{C}} \ldots (f_{\alpha_N}^N, g_{\beta_N}^N)_{\mathbb{C}}. \tag{17b}$$

Finally we note that our definition (15) of the QHS tensor product applies also to the case if some Hilbert spaces in the product (15) have the 'reduced' quaternionic structure (i.e. are $\mathrm{SU}(2)_c$ singlets and carry an abelian representation of the quaternionic group).

4. Generalisation

Now for the further applications we generalise the concept of the QHS. Because the QHS formalism admits also the complex Hilbert spaces with reduced quaternionic structure, therefore we should consider the direct sum of the QHS and the complex space†. In such sums, vectors from the quaternionic sector are multiplied by quaternions from $\mathbb{Q}$ whereas vectors from the complex one are multiplied by complex numbers from $\mathbb{C} \subset \mathbb{Q}$. Thus we have to deal with a Hilbert module rather than the Hilbert space, defined over the ring‡ $(\mathbb{Q}, \mathbb{C})$. Instead of the Hilbert module we can speak of the complex Hilbert space with suitable structure, namely the underlying space of the direct sum of one- and two-dimensional representations of the $\mathrm{SU}(2)_c$ group. Note that in the Hilbert module under consideration it can be possible to introduce only complex (or real) non-degenerate geometry. This follows from the fact that the ring $(\mathbb{Q}, \mathbb{C})$ contains only complex or real fields ($(\mathbb{C}, \mathbb{C})$ or $(\mathbb{R}, \mathbb{R})$). In the space introduced above we can consider an invariance group G of the complex scalar product containing $\mathrm{U}(2)_c \supset \mathrm{SU}(2)_c$ as the subgroup. The group G will be identified with a symmetry (eventually broken) of the physical theory. For consistency we must demand that the product space is the underlying one for this symmetry group G. It is obvious that to be in agreement with

† Direct sum of modules is discussed in Bourbaki (1974).

‡ The elements of the $(\mathbb{Q}, \mathbb{C})$ ring have the form $\begin{pmatrix} q & 0 \\ 0 & \alpha \end{pmatrix}$, $q \in \mathbb{Q}$, $\alpha \in \mathbb{C}$.

definition (15) of the QHS tensor product, the admissible representations $D(\mathrm{G})$ of G can contain the singlets and doublets of $\mathrm{SU}(2)_c$ only, namely

$$D(\mathrm{G})\downarrow \mathrm{SU}(2)_c = (\oplus\mathbf{2})\oplus(\oplus\mathbf{1}). \tag{18}$$

Thus the definition (15) of the QHS tensor product should be rewritten in the form

$$\mathscr{H}^1\times\mathscr{H}^2\times\ldots\times\mathscr{H}^N=\Pi(\mathscr{H}^1\underset{c}{\otimes}\mathscr{H}^2\underset{c}{\otimes}\ldots\underset{c}{\otimes}\mathscr{H}^N) \tag{19}$$

where the $\mathscr{H}^k$ are the underlying spaces of the admissible representations of G and Π project on the admissible subspace in the standard complex tensor product ($\underset{c}{\otimes}$) of the $\mathscr{H}^k$. In Rembieliński (1980a, b) the classification problem of the admissible groups and their representations was analysed. It was found that the linearly realised group $\mathrm{G}\supset\mathrm{U}(2)_c$ must be of the form $\mathrm{G}=\mathrm{G}_\mathrm{F}\times\mathrm{G}_c$ where G_c is simple and $\mathrm{SU}(2)_c\subset\mathrm{G}_c$. If some reasonable physical conditions hold (like the validity of the quark hypothesis) then the only admissible G_c are the special unitary groups $\mathrm{SU}(3r)_c$ where r is odd. Furthermore, it was shown that the set of admissible representations of $\mathrm{SU}(3r)_c$ contains only scalar representations, a $\binom{3r}{r}$-dimensional one and its conjugate. For $r=1$ i.e. for $\mathrm{G}_c=\mathrm{SU}(3)_c$ the admissible representations (except the trivial one) coincide with the basic representations $\mathbf{3}$ and $\mathbf{3}^*$.

The class of the admissible Lie groups can be extended if we admit the nonlinear realisations. As is well known, nonlinear realisations of a group G are induced by the so-called stability subgroup H which is realised linearly (Coleman *et al* 1969, Isham 1969, Salam and Strathdee 1969). This denotes that the nonlinear (irreducible) realisation of G is determined by a linear (irreducible) representation of H. Therefore for $\mathrm{G}\supset\mathrm{H}=\mathrm{G}_\mathrm{F}\times\mathrm{SU}(3r)_c$ the nonlinear realisations of G induced by admissible representations of $\mathrm{SU}(3r)_c$ are admissible too.

5. Observable and unobservable states. Observables and symmetries

This section is devoted to some physical consequences of QHS formalism.

5.1. States

As was mentioned in § 3, pure states are in our formalism associated with complex rays from QHS. Therefore, we must formulate the quantum mechanical notions (like probability transitions, observables etc) within the complex geometry framework. In Rembieliński (1980a, b) it was shown that the observable states (i.e. states of an observable physical system) can be eventually associated with the $\mathrm{SU}(2)_c$ singlets only. In fact, from equation (16*a*) it follows that the product state constructed from two doublets f^1 and f^2 has the form $\chi_{\alpha\beta}=(1/\sqrt{2})(f^1_\alpha f^2_\beta-f^1_\beta f^2_\alpha)$ and thus the two-particle state $\chi_{\alpha\alpha}$ is equal to zero irrespective of the detailed specification of f^1 and f^2. This feature is inadmissible for observable systems because a physical *system* composed from observable *subsystems* should also be observable i.e. the set of its possible *states* should be non-empty. The above argument has the following motivation. The notion of the physical *system* is equivalent on the quantum mechanical ground to the 'set of all *states* of this system' or in the group theoretical language, for elementary systems, to the 'set of all states obtained by action of the space–time symmetry group (Poincaré, Galilean . . .) on a fixed state of the system (on the state of the system at rest, for example)' i.e. an elementary system is represented by the orbit of the space–time group in the Hilbert space of states† (Barut and Rączka 1977). A *system* formed from at least two *observable subsystems* (particles) must occur in bound and/or scattering *observable states*‡ because it carries non-vanishing observable quantum numbers such as the four momentum§ for example. Therefore the Aristotelian law of the excluded middle implies that non-existence of the two-component *system* (→ the set of its possible states is empty) formed from two copies of an elementary *subsystem* must be interpreted as unobservability of the latter i.e. it can eventually occur in reality in the composed systems only (other than the two-component mentioned above).

Note that the above mechanism of confinement of f remembers the Pauli principle. However for *observable fermions*, the two-fermion *system* is also observable because the set of possible two-fermion *states* is non-empty: only some *exceptional* two-fermion *states* are forbidden by the Pauli principle. For this reason our confinement mechanism does not contradict the Pauli exclusion principle.

In conclusion, the doublets of $SU(2)_c$ cannot be identified with observable states. Because the QHS formalism admits $SU(2)_c$ singlets and doublets only, the observable states should be associated with $SU(2)_c$ singlets. For admissible $SU(3r)_c$ groups this fact implies the analogous property (Rembieliński 1980a, b): $SU(3r)_c$ singlets only are observable whereas the admissible $\binom{3r}{r}$-dimensional multiplets must be associated with unobservable particles. Because the $\binom{3r}{r}$-dimensional representation does not coincide with the $(3r)^2-1$-dimensional adjoint representation of $SU(3r)_c$, this multiplet should be identified with quarks. Consequently, because of the unobservability of the $SU(3r)_c$ degrees of freedom this group should be identified with the colour group‖. Note that for $SU(3r)_c$ singlets (observable states) the tensor multiplication rule (19) reduces to the standard one.

A question arises: how to distinguish between the possible candidates for physical colour group. At least for three reasons the $SU(3)_c$ group is preferred:

(i) Only for $SU(3)_c$ does the quark multiplet span the self-representation space $\mathbf{3}(\mathbf{3}^*)$ of the colour group.

(ii) The above feature minimalises the number of colours to three. In the general $SU(3r)_c$ case this number equals the dimensionality of the quark multiplet: $n = (3r)!/r!(2r)!$ i.e. for $r>1$, $n>84$.

(ii) If we take into account the flavour degrees of freedom then the parameter $R=[\sigma(e^+e^-\to \text{hadrons})]/[\sigma(e^+e^-\to\mu^+\mu^-)]$ becomes $R(r)=n(r)\Sigma(q_{\text{flavour}})^2$. Thus for four (u, d, s, c) flavours $R(r)=n(r)\frac{10}{9}$ i.e. for $r>1$ we obtain the unacceptable value $R>93$.

5.2. *Observables*

As was mentioned above, the quantum mechanical notions should be formulated within the complex geometry framework. Therefore we associate the physical observables with the $\mathbb{C}$-self-adjoint operators. Moreover we demand that in the observable sector

† Analogous to classical mechanics where a mechanical system is represented by all points of its phase space.
‡ In the field theory framework, observability of scattering states is exemplified by the clustering property of the Wightman functions.
§ This follows from observability of subsystems and the additivity of the four momentum. Note that this composed system should at least interact with the gravitational field.
‖ It is interesting that $SU(3r)_c$, r odd, was proposed by Kingsley (see Greenberg and Nelson 1977) as an alternative way of introducing colour degrees of freedom.

of the space of states the laws of ordinary quantum mechanics hold. This is consistent with definition (19) of the tensor product of the QHS. On the other hand, the existence of the unobservable states implies the superselection rule: the matrix elements of observables between observable and unobservable states must vanish. This is equivalent to the non-existence of the physical superposition of the observable and unobservable states. From the group theoretical point of view, this superselection rule is generated by conservation of 'triality'. In fact, the $\binom{3r}{r}$-dimensional admissible representations of $SU(3r)_c$ (associated with unobservable states) are contained in the expansion of the direct product of r self-representations of this group. Because the centre of $SU(3r)_c$ is composed of elements of the form $\exp(2ki\pi/3r)$ (i.e. it is isomorphic to the cyclic group Z_{3r}) then the centre of such an admissible representation contains only elements of the form $[\exp(2ki\pi/3r)]^r = \exp(2ki\pi/3)$ i.e. it is isomorphic to Z_3. Thus with every admissible representation we can associate its triality conserved modulo three: triality 0 with the singlet of $SU(3r)_c$, triality ± 1 with two mutually conjugate $\binom{3r}{r}$-dimensional representations. So the observable states have triality zero. The triality operator must commute with every observable and every canonical transformation. As the consequence of the above discussion the average value $\langle\Omega\rangle$ of an observable Ω in a state ψ has the physical sense only in the observable sector of $\mathscr{H}$.

At this point it is sensible to remark that the idea of the split space of states is nothing new—it is assumed in each theory with the Pauli-like principle: observable states = colour singlets. Octonionic theories (see § 1) are constructions of this sort for example. On the dynamical level this question arises in quantum chromodynamics.

5.3. Symmetries

Of great importance for the physical applications of the QHS scheme is the symmetry problem. It is reasonable to demand that each *exact* symmetry group G_0 of the physical theory should be realised in the space of states† as a set of transformations with both active and passive interpretation, at least on mathematical grounds, and which does not destroy the algebraical and geometrical structure of $\mathscr{H}$. As was mentioned in § 4 a symmetry group G (eventually broken) must have the form of a direct product $G = G_F \times G_c$ for linearly realised G, or $G \supset G_F \times G_c$ with $G/G_F \times G_c$ acting nonlinearly‡. Because the nonlinear transformations cannot correspond to an exact symmetry but rather to at least a spontaneously broken one (Coleman *et al* 1969, Salam and Strathdee 1969) then G_0 must belong to $G_F \times G_c$. Obviously the flavour group G_F does not destroy the quaternionic structure of $\mathscr{H}$ and the question of an exact symmetry subgroup of G_F is determined fully by the dynamics§. Therefore we restrict our attention to the colour group G_c. Firstly, we note that the group $SU(2)_c \subset G_c$ is realised in the $\mathbb{Q}$-sector of the generalised QHS as $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$-unitary operations (see §§ 2.7, 3, 4) and as the identity in the $\mathbb{C}$-sector. Thus it leaves the full geometrical structure of the QHS unaffected. However, in general, $SU(2)_c$ transformations can be interpreted only as the active ones. Really, the $SU(2)_c$ transformations of the

† Because in our formalism pure states are represented by complex rays and there exists the superselection rule, the space of states does not coincide with $\mathscr{H}$ but rather is the direct sum of the coherent projective manifolds.

‡ But rather in the ring of the field operators than in the space of states.

§ Note, however, that in a homomorphic realisation of $G_F \times G_c$ few elements from the centre of G_F and G_c can be identified as for example in the case $U(1) \times SU(N) \to U(N)$ by the identification of the elements $\exp(i\pi k/N)$ in U(1) and SU(N). Such a situation can have physical implications.

components f_α of a vector $f = e_\alpha f_\alpha$ cannot, in general, be compensated by transformations of the quaternionic units e_μ which must belong† to SO(3) (i.e. to the automorphism group of $\mathbb{Q}$)—otherwise they break the algebraic structure of the QHS. However, if we take into account transformations $U(2)_c \supset SU(2)_c$ then we see that the action of the form‡

$$\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & e^{e_3\varphi} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

can be compensated by the transformation

$$(e_0, e_1, e_2, e_3) \to (e_0, e_1 \cos\varphi + e_2 \sin\varphi, e_2 \cos\varphi - e_1 \sin\varphi, e_3)$$

(belonging to SO(3)) of the quaternionic units. In addition, the phase transformations from $U(1)_0 = \{\exp(\lambda e_3)\sigma_0\}$ do not change the states (complex rays). Summarising, only the transformations of the form

$$e^{e_3\lambda} \begin{pmatrix} 1 & 0 \\ 0 & e^{e_3\varphi} \end{pmatrix}$$

belonging to $U(1)_0 \times U(1)_c \subset U(2)_c$ can be interpreted as the exact symmetry because they are $\mathbb{C}$-unitary§, do not change the algebraical and geometrical structure of QHS together with the tensor multiplication rules and have both active and passive implementation in the space of states ($\mathbb{C}$-projective space).

A question arises: how to interpret the other transformations of $U(2)_c$. To do this let us note that the automorphism group SO(3) of $\mathbb{Q}$ is isomorphic to U(2)/U(1). Thus with each transformation

$$\begin{pmatrix} f_0' \\ f_1' \end{pmatrix} = U \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \tag{20a}$$

where $U \in U(2)_c$ we can associate a suitable automorphism $R(U) \in SO(3)$ of $\mathbb{Q}$

$$e_0' = e_0 \qquad e' = Re \tag{20b}$$

under the condition that $U(1)_c$ coincides with the subgroup $SO(2)_c \subset SO(3)$ of the rotations about the e_3 axis in $\mathbb{Q}$. Now, because the transformations from $U(2)_c/U(1)_0 \times U(1)_c \sim SO(3)/SO(2)_c$ cannot be identified with an exact symmetry, they can be interpreted as passing to the unitary non-equivalent but isomorphic QHS. This set of unitary non-equivalent QHS is globally homeomorphic to the homogeneous space $SO(3)/SO(2)_c$ i.e. to the surface of a sphere. From this point of view, the choice of a concrete QHS is the convention connected with the choice of a concrete basis‖ $\{e_\mu\}$ in the quaternion field $\mathbb{Q}$. Such a situation can be interpreted on field theory grounds as the spontaneous breaking¶ (Kuo 1971a, b) of $U(2)_c$ to the subgroup $U(1)_0 \times U(1)_c$.

† The components f_α form doublets of $SU(2)_c$ whereas the quaternionic units e_0 and e form a singlet and triplet respectively.

‡ The phase transformations $f_\alpha \to f_\alpha \exp(\lambda e_3)$, $\lambda \in \mathbb{R}$, are $\mathbb{R}$ and $\mathbb{C}$ unitary ($\mathbb{C}$ unitarity is important from the physical point of view) but are not $\mathbb{Q}$ unitary. Nevertheless they do not change the $\mathbb{Q}$ geometrical structure of the QHS because $\mathbb{Q}$-rays and the length of the vectors remain unaffected.

§ Note that the automorphisms of $\mathbb{Q}$ belonging to $SO(3)/SO(2)_c$ are not implemented by $\mathbb{C}$ unitary (antiunitary) operators in QHS (for fixed $\mathbb{C}$-solar product of course).

‖ More precisely with the choice of a concrete direction of the third axis (e_3) in the quaternion algebra.

¶ In his paper Kuo (1971) investigated the relation between convention in quantum theories and broken symmetries. However his claim that the outer automorphisms of an exact symmetry group should be exact symmetries themselves is in my opinion rather questionable.

Now let us extend our consideration to the physically preferable $G_c = SU(3)_c \supset SU(2)_c$ case. Because the generalised QHS splits on the $SU(3)_c$ triplet (anti-triplet) and singlet sectors, it is sufficient to restrict ourselves to the (unobservable) triplet one. Every vector from such a subspace has the form

$$\psi = \sum_{\alpha=0}^{1} \varepsilon_\alpha f_\alpha(e_0, e_3) + \epsilon_0 \chi(e_0, e_3)$$

where the $SU(3)_c$ triplet (f_α, χ) contains the doublet f_α and singlet χ of $SU(2)_c$ while ε_μ and ϵ_λ form the basis in the ring $(\mathbb{Q}, \mathbb{C})$ (see § 4), namely

$$\varepsilon_0 = \begin{pmatrix} e_0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \boldsymbol{e} & 0 \\ 0 & 0 \end{pmatrix} \qquad \epsilon_0 = \begin{pmatrix} 0 & 0 \\ 0 & e_0 \end{pmatrix} \qquad \epsilon_3 = \begin{pmatrix} 0 & 0 \\ 0 & e_3 \end{pmatrix}.$$

The $\mathbb{C}$-scalar product of two vectors $\psi = (f_\alpha, \chi)$ and $\varphi = (g_\alpha, \xi)$ reads

$$(\psi, \varphi)_{\mathbb{C}} = \sum_{\alpha=0}^{1} (f_\alpha, g_\alpha)_{\mathbb{C}} + (\chi, \xi)_{\mathbb{C}}.$$

The automorphism group of the ring $(\mathbb{Q}, \mathbb{C})$ is the direct product $SO(3) \times Z_2$ where $SO(3)$ acts in $\mathbb{Q}$ while the non-trivial element of the cyclic subgroup Z_2 multiplies ϵ_3 by -1. Now, it is easy to verify that only transformations of the form

$$\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & e^{e_3 \varphi} & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}$$

belonging to $U(3) \supset SU(3)_c$ are compensated by suitable automorphisms from $SO(3) \times Z_2$. Taking into account that the phase transformations from $U(1)_0$ leave states invariant, we conclude that the exact symmetry form in this case is the $U(1) \times U(1)$ group which has the following elements (in the complex language)

$$\begin{pmatrix} e^{i\alpha} & \cdot & \cdot \\ \cdot & e^{i\beta} & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix}.$$

The common subgroup of $U(1) \times U(1)$ and $SU(3)_c$ contains elements of the form

$$\begin{pmatrix} e^{i\varphi} & \cdot & \cdot \\ \cdot & e^{-2i\varphi} & \cdot \\ \cdot & \cdot & e^{i\varphi} \end{pmatrix}$$

generated by the element

$$\begin{pmatrix} \frac{1}{3} & \cdot & \cdot \\ \cdot & -\frac{2}{3} & \cdot \\ \cdot & \cdot & \frac{1}{3} \end{pmatrix}$$

of the Lie algebra of $SU(3)_c$. Now, we note that the transformations from $U(3)/U(2)_c$ break the algebraic structure of generalised QHS because there is not a homomorphism from $U(3)/U(2)_c$ to the automorphism group $SO(3)\times Z_2$ of the ring $(\mathbb{Q}, \mathbb{C})$. Thus the symmetry breaking holds in two steps: firstly† to $U(2)_c$ and secondly to $U(1)\times U(1)$. At this point it is sensible to remark that only $SU(3)_c$ colour degrees of freedom are algebraically confined: the generator of phase transformations should be associated with an unconfined charge like the baryonic number. It is interesting that in the field theory context, after gauging $SU(3)_c$ we have as usual the octet of gauge fields (gluons) but only one, that associated with the exact subgroup of $SU(3)_c$, remains massless (coloured 'photon') while seven acquire masses because of the symmetry breaking. Thus in this case gauging of the colour $SU(3)_c$ does not lead to the infrared problem (see, for example, Gross 1976). It may be surprising that we deal with the inadmissible eight-dimensional representation of $SU(3)_c$. However, in our formalism only *states* from QHS cannot form inadmissible multiplets while *fields* must appear in admissible products only. For this reason the gluon fields appear in QHS theory in admissible products, for example, with the quark fields via the covariant derivatives. On the other hand, the gluonic quanta are absent in the space of states (except eventual admissible bound states). Finally, we note that in the case of explicit symmetry breaking via the Higgs mechanism, the Higgs field must form a non-trivial representation of $SU(3)_c$ and therefore the Higgs quanta cannot be observable.

6. Conclusions

Let us summarise the results. Our starting point consisted of the assumption of the quaternionic structure of the space of states. We have shown that the pure states should be identified with the complex rays in QHS. As a consequence of these two facts we have obtained the suitable form of the QHS tensor product (equation (15)). The appearance of a larger symmetry $G \supset SU(2)_c$ leads to the suitable G_c-dependence of the QHS tensor product (equation (19)). The only admissible‡ symmetry groups have the form $G_F \times SU(3r)_c$, r odd. Furthermore the degrees of freedom of $SU(3r)_c$ are algebraically confined. This is implied by the tensor multiplication rules§. For physical reasons the group $SU(3)_c$ is preferred. It is broken down to the one-parameter subgroup. This causes the absence of the infrared problem in QHS theory.

The other topics connected with QHS theory like the unification problem of strong and electro-weak interactions and investigation of a concrete dynamical model in this framework will be done elsewhere.

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I should like to thank P Kosiński, M Majewski and W Tybor for interesting discussions.

† In this case the $U(2)_c$ transformations have the form $e^{i\alpha}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ where $u \subset SU(2)_c$.

‡ Admissible for two reasons: mathematical and physical (§ 4).

§ Such a situation is nothing new in quantum physics. The tensor multiplication rules for the bosons or fermions are very similar: to obtain the space of states one must project the tensor product space on the subspace of the fully symmetrised or antisymmetrised wavefunctions. In other words the other than one-dimensional (fully symmetrical and antisymmetrical) representations of the permutation group are inadmissible.

Appendix. The scalar products in the quaternion algebra

The quaternionic scalar product: $\langle a, b\rangle_{\mathbb{Q}} = \bar{a}b \in \mathbb{Q}$. The complex scalar product: $\langle a, b\rangle_{\mathbb{C}} = \frac{1}{2}(\bar{a}b + \widetilde{\bar{a}b}) = A^*_\alpha B_\alpha \in \mathbb{C}$. The real scalar product: $\langle a, b\rangle_{\mathbb{R}} = \frac{1}{2}(\bar{a}b + \overline{\bar{a}b}) = a_\mu b_\mu \in \mathbb{R}$. Here $a, b \in \mathbb{Q}$, $a = e_\mu a_\mu = e_\alpha A_\alpha$, $\mu = 0, 1, 2, 3$, $\alpha = 0, 1$. The properties:

$$\langle a, b + c\rangle_{\mathbb{A}} = \langle a, b\rangle_{\mathbb{A}} + \langle a, c\rangle_{\mathbb{A}} \qquad \langle a, b\rangle_{\mathbb{A}}^{C_{\mathbb{A}}} = \langle b, a\rangle_{\mathbb{A}}$$

$$\langle a, ab\rangle_{\mathbb{A}} = \tfrac{1}{2}(b + b^{C_{\mathbb{A}}})|a|^2 \qquad \langle a, b\alpha\rangle_{\mathbb{A}} = \langle a, b\rangle_{\mathbb{A}}\alpha \qquad \text{for } \alpha \in \mathbb{A}.$$

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